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Research Article

Picard Sequence and Fixed Point Results on *b*-Metric Spaces

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We obtain some fixed point results for single-valued and multivalued mappings in the setting of a b-metric space. These results are generalizations of the analogous ones recently proved by Khojasteh, Abbas, and Costache.

1. Introduction

It is well known that the contraction mapping principle, formulated and proved in the Ph.D. dissertation of Banach in 1920, which was published in 1922 [1], is one of the most important theorems in classical functional analysis. It is widely considered as the source of metric fixed point theory. Also its significance lies in its vast applicability in a number of branches of mathematics. The study of fixed points of mappings satisfying a certain metrical contractive condition attracted many researchers; see, for example, [2–10].

The concept of b-metric space appeared in some works, such as Bakhtin [11] and Czerwik [12, 13]. In dealing with the fixed point theory for single-valued and multivalued mappings in b-metric spaces, seven years later, Khamsi [14] and Khamsi and Hussain [15] reintroduced such spaces under the name of metric-type spaces. For some results of fixed and common fixed point in the setting of b-metric spaces, see [3, 14–18].

In the first part of this paper we consider a class of Picard sequences with a property of regularity and prove that these sequences are Cauchy. Then we use this result for establishing existence of fixed points for single-valued mappings. The last part contains a fixed point result for multivalued mappings on complete b-metric spaces. Clearly, these results are a generalization of the results recently obtained from Khojasteh et al. in [5].

2. Preliminaries

The aim of this section is to collect some relevant definitions and results for our further use. Let $\mathbb{N}$ be the set of positive integers, $\mathbb{R}$ the set of real numbers, and $\mathbb{R}_+$ the set of nonnegative real numbers.

Definition 1. Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}_+$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

A triplet (X, d, s) is called a b-metric space.

We observe that a b-metric space is a metric space if $s = 1$. So the notions of convergent sequence, Cauchy sequence, and complete space are defined as in metric spaces.

Next, we give some examples of b-metric spaces.

Example 2. Let $X = [0, 1]$ and $d : X \times X \to \mathbb{R}_+$ defined by $d(x, y) = (x - y)^2$, for each $x, y \in X$. Clearly, $(X, d, 2)$ is a b-metric space.

Example 3. Let $C_b(X) = \{f : X \to \mathbb{R} : \|f\|_\infty = \sup_{x\in X}|f(x)| < +\infty\}$ and let $\|f\| = \sqrt[3]{\|f^3\|_\infty}$. The function $d : C_b(X) \times C_b(X) \to \mathbb{R}_+$ defined by

$$d(f,g) = \|f-g\|, \quad \forall f,g \in C_b(X) \tag{1}$$

is a b-metric with constant $s = \sqrt[3]{4}$ and so $(C_b(X), d, \sqrt[3]{4})$ is a b-metric space.

We note that if a, b are two nonnegative real numbers, then

$$(a+b)^3 \le 4\left(a^3+b^3\right), \qquad \sqrt[3]{a+b} \le \sqrt[3]{a} + \sqrt[3]{b}. \tag{2}$$

This implies that

$$d(f,g) \le \sqrt[3]{4}\,(d(f,h)+d(h,g)), \quad \forall f,g,h \in C_b(X). \tag{3}$$

The following results are useful for some of the proofs in the paper.

Lemma 4. *Let (X,d,s) be a b-metric space and let $\{x_n\}$ be a sequence in X. If $x_n \to y$ and $x_n \to z$, then $y = z$.*

Lemma 5. *Let (X,d,s) be a b-metric space and let $\{x_k\}_{k=0}^n \subset X$. Then*

$$\begin{aligned} d(x_n,x_0) \le{}& sd(x_0,x_1) + \cdots + s^{n-1}d(x_{n-2},x_{n-1}) \\ &+ s^{n-1}d(x_{n-1},x_n). \end{aligned} \tag{4}$$

From Lemma 5, we deduce the following lemma.

Lemma 6. *Let $\{x_n\}$ be a sequence in a b-metric space (X,d,s) such that*

$$d(x_n,x_{n+1}) \le \lambda d(x_{n-1},x_n) \tag{5}$$

for some λ, $0 < \lambda < 1/s$, and each $n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence in X.

Let (X,d,s) be a b-metric space and let $CB(X)$ be the collection of all nonempty closed bounded subsets of X. For $A, B \in CB(X)$, define

$$H(A,B) = \max\{\delta(A,B), \delta(B,A)\}, \tag{6}$$

where

$$\begin{aligned} \delta(A,B) &= \sup\{d(a,B) : a \in A\}, \\ \delta(B,A) &= \sup\{d(b,A) : b \in B\} \end{aligned} \tag{7}$$

with

$$d(a,C) = \inf\{d(a,x) : x \in C\}. \tag{8}$$

Note that H is called the Hausdorff b-metric induced by the b-metric d.

We recall the following properties from Czerwik et al. [12, 13, 19].

Lemma 7. *Let (X,d,s) be a b-metric space. For any $A, B, C \in CB(X)$ and any $x, y \in X$, we have the following:*

(i) *$d(x,B) \le d(x,b)$, for any $b \in B$;*

(ii) *$\delta(A,B) \le H(A,B)$;*

(iii) *$d(x,B) \le H(A,B)$, for any $x \in A$;*

(iv) *$H(A,A) = 0$;*

(v) *$H(A,B) = H(B,A)$;*

(vi) *$H(A,C) \le s[H(A,B) + H(B,C)]$;*

(vii) *$d(x,A) \le s[d(x,y) + d(y,A)]$.*

Lemma 8. *Let (X,d,s) be a b-metric space and $A, B \in CB(X)$ with $H(A,B) > 0$. Then for each $h > 1$ and for all $b \in B$ there exists $a \in A$ such that $d(a,b) \le hH(A,B)$.*

Lemma 9. *Let (X,d,s) be a b-metric space. For $A \in CB(X)$ and $x \in X$, we have*

$$d(x,A) = 0 \Longleftrightarrow x \in \overline{A} = A, \tag{9}$$

where $\overline{A}$ denotes the closure of the set A.

3. Picard Sequence in b-Metric Spaces

Let (X,d,s) be a b-metric space, let $x_0 \in X$, and let $f : X \to X$ be a given mapping. The sequence $\{x_n\}$ with $x_n = f^n x_0 = fx_{n-1}$ for all $n \in \mathbb{N}$ is called a Picard sequence of initial point x_0. In this section we consider a class of Picard sequences which are Cauchy.

Proposition 10. *Let (X,d,s) be a b-metric space and let $f : X \to X$ be a given mapping. Assume that a Picard sequence $\{x_n\}$ of initial point $x_0 \in X$ satisfies one of the following conditions:*

$$d(x_n,x_{n+1}) \le \frac{d(x_{n-1},x_n) + d(x_n,x_{n+1})}{d(x_{n-1},x_n) + d(x_n,x_{n+1}) + 1}\, d(x_{n-1},x_n) \tag{10}$$

or

$$\begin{aligned} d(x_n,x_{n+1}) \le{}& \left(\frac{d(x_{n-1},x_n) + d(x_n,x_{n+1})}{d(x_{n-1},x_n) + d(x_n,x_{n+1}) + a}\right)^{1/2} \\ &\cdot d(x_{n-1},x_n), \end{aligned} \tag{11}$$

where a is a positive real number. Then $\{x_n\}$ is a Cauchy sequence.

Proof. Let $x_0 \in X$ be an arbitrary point, and let $\{x_n\}$ be a Picard sequence of initial point x_0. If $x_{n_0} = x_{n_0-1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of f and so $\{x_n\}$ is a Cauchy sequence.

Assume that (10) holds for the sequence $\{x_n\}$. If $x_n \ne x_{n-1}$ for all $n \in \mathbb{N}$, from (10), we deduce that the sequence $\{d(x_{n-1},x_n)\}$ is decreasing. Thus there exists a nonnegative real number r such that $d(x_{n-1},x_n) \to r$. Then we claim that

$r = 0$. If $r > 0$, on taking limit as $n \to +\infty$ on both sides of (10), we obtain

$$r \leq \frac{r+r}{r+r+1} r < r \tag{12}$$

which is a contradiction. It follows that $r = 0$.

Now, we prove that $\{x_n\}$ is a Cauchy sequence. Let $\lambda \in [0, s^{-1}[$. Since $r = 0$, then there exists $n(\lambda) \in \mathbb{N}$ such that

$$\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} \leq \lambda, \quad \forall n \geq n(\lambda). \tag{13}$$

This implies that

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n), \quad \forall n \geq n(\lambda) \tag{14}$$

and by Lemma 6 we deduce again that $\{x_n\}$ is a Cauchy sequence.

The proof that $\{x_n\}$ is a Cauchy sequence if (11) holds is the same. □

As a consequence of the previous result, we establish the following result of existence of fixed points.

Theorem 11. *Let (X, d, s) be a complete b-metric space and let $f : X \to X$ be a mapping such that*

$$sd(fx, fy) \leq \frac{d(x, fy) + d(y, fx)}{d(x, fx) + d(y, fy) + 1} d(x, y) \tag{15}$$

for all $x, y \in X$. Then

(i) *f has at least one fixed point $z \in X$;*

(ii) *every Picard sequence of initial point $x_0 \in X$ converges to a fixed point of f;*

(iii) *if $z, w \in X$ are two distinct fixed points of f, then $d(z, w) \geq s/2$.*

Proof. Let $x_0 \in X$ be an arbitrary point and let $\{x_n\}$ be a Picard sequence of initial point x_0. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of f. If $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$, using the contractive condition (15) with $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned} sd(x_n, x_{n+1}) & \\ &\leq \frac{d(x_{n-1}, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} d(x_{n-1}, x_n) \\ &\leq \frac{s[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} d(x_{n-1}, x_n); \end{aligned} \tag{16}$$

that is, condition (10) holds for the sequence $\{x_n\}$. Then, by Proposition 10, $\{x_n\}$ is a Cauchy sequence. Since X is a complete b-metric space, the sequence $\{x_n\}$ converges to some $z \in X$. Now, we prove that z is a fixed point of f. Using (15) with $x = x_n$ and $y = z$, we obtain

$$\begin{aligned} sd(x_{n+1}, fz) &= sd(fx_n, fz) \\ &\leq \frac{d(x_n, fz) + d(z, fx_n)}{d(x_n, fx_n) + d(z, fz) + 1} d(x_n, z) \\ &= \frac{d(x_n, fz) + d(z, x_{n+1})}{d(x_n, x_{n+1}) + d(z, fz) + 1} d(x_n, z). \end{aligned} \tag{17}$$

Moreover, from

$$\begin{aligned} d(z, fz) - sd(x_n, z) &\leq sd(x_n, fz) \\ &\leq s^2 [d(x_n, z) + d(z, fz)] \end{aligned} \tag{18}$$

as $n \to +\infty$, we deduce that

$$\begin{aligned} d(z, fz) &\leq \liminf sd(x_n, fz) \\ &\leq \limsup sd(x_n, fz) \leq s^2 d(z, fz). \end{aligned} \tag{19}$$

On taking lim inf, as $n \to +\infty$, on both sides of (17), by (19) we get

$$\begin{aligned} d(z, fz) &\leq \liminf sd(x_{n+1}, fz) \\ &\leq \frac{s^2\, d(z, fz)}{d(z, fz) + 1} \limsup d(x_n, z) = 0. \end{aligned} \tag{20}$$

This implies that $d(z, fz) = 0$; that is, $z = fz$ and hence z is a fixed point of f. Thus (i) and (ii) hold.

If $w \in X$, with $z \neq w$, is another fixed point of f, then using (15) with $x = z$ and $y = w$, we get

$$\begin{aligned} sd(z, w) &= sd(fz, fw) \\ &\leq [d(z, fw) + d(w, fz)]\, d(z, w) \\ &= 2d^2(z, w) \end{aligned} \tag{21}$$

and hence $d(z, w) \geq s/2$; that is, (iii) holds. □

Remark 12. From Theorem 11, we obtain Theorem 1 of [5] if $s = 1$; that is, if (X, d, s) is a metric space.

In the following result we consider a weak contractive condition.

Theorem 13. *Let (X, d, s) be a complete b-metric space and let $f : X \to X$ be a mapping such that*

$$\begin{aligned} sd(fx, fy) \leq{} & \frac{d(x, fy) + d(y, fx)}{d(x, fx) + d(y, fy) + 1} d(x, y) \\ & + Ld(y, fx) \end{aligned} \tag{22}$$

for all $x, y \in X$, where L is a nonnegative real number. Then

(i) *f has at least one fixed point $z \in X$;*

(ii) *every Picard sequence of initial point $x_0 \in X$ converges to a fixed point of f;*

(iii) *if $z, w \in X$ are two distinct fixed points of f, then $d(z,w) \geq \max\{(s-L)/2, 0\}$.*

Proof. Let $x_0 \in X$ be an arbitrary point and let $\{x_n\}$ be a Picard sequence of initial point x_0. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of f. If $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$, using the contractive condition (22) with $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned} s d(x_n, x_{n+1}) &\leq \frac{d(x_{n-1}, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} d(x_{n-1}, x_n) \\ &\quad + L d(x_n, x_n) \\ &\leq \frac{s[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} d(x_{n-1}, x_n); \end{aligned} \tag{23}$$

that is, condition (10) holds for the sequence $\{x_n\}$. Then, by Proposition 10, $\{x_n\}$ is a Cauchy sequence. Since X is a complete b-metric space, the sequence $\{x_n\}$ converges to some $z \in X$. Now, we prove that z is a fixed point for f. Using (22) with $x = x_n$ and $y = z$, we obtain

$$\begin{aligned} s d(x_{n+1}, fz) &= s d(f x_n, fz) \\ &\leq \frac{d(x_n, fz) + d(z, f x_n)}{d(x_n, f x_n) + d(z, fz) + 1} d(x_n, z) + L d(z, f x_n) \\ &= \frac{d(x_n, fz) + d(z, x_{n+1})}{d(x_n, x_{n+1}) + d(z, fz) + 1} d(x_n, z) + L d(z, x_{n+1}). \end{aligned} \tag{24}$$

On taking lim inf as $n \to +\infty$ on both sides of (24), by (19) we get

$$\begin{aligned} d(z, fz) &\leq \liminf s d(x_{n+1}, fz) \\ &\leq \frac{s^2 d(z, fz)}{d(z, fz) + 1} \limsup d(x_n, z) \\ &\quad + \liminf L d(z, x_{n+1}) = 0. \end{aligned} \tag{25}$$

This implies that $d(z, fz) = 0$; that is, $z = fz$ and hence z is a fixed point of f. Thus (i) and (ii) hold.

If $w \in X$, with $z \neq w$, is another fixed point of f, then using (22) with $x = z$ and $y = w$, we get

$$\begin{aligned} s d(z, w) &= s d(fz, fw) \\ &\leq [d(z, fw) + d(w, fz)] d(z, w) + L d(w, fz) \\ &= 2 d^2(z, w) + L d(z, w) \end{aligned} \tag{26}$$

and hence $s \leq 2d(z, w) + L$; that is, (iii) holds. □

Next examples illustrate Theorem 11.

Example 14. Let $X = \{\alpha, \beta, \gamma, \delta\}$ and $d : X \times X \to \mathbb{R}_+$ defined by

$$\begin{aligned} &d(\alpha, \beta) = d(\alpha, \delta) = d(\gamma, \delta) = 2, \\ &d(\alpha, \gamma) = d(\beta, \delta) = 3, \\ &d(\beta, \gamma) = 6, \\ &d(x, y) = d(y, x), \qquad d(x, x) = 0 \\ &\qquad\qquad\qquad \forall x, y \in X. \end{aligned} \tag{27}$$

Clearly, $(X, d, 6/5)$ is a complete b-metric space. Consider the mapping $f : X \to X$ defined by

$$fx = \begin{cases} \alpha & \text{if } x \in \{\alpha, \gamma, \delta\} \\ \beta & \text{if } x = \beta. \end{cases} \tag{28}$$

Now, we have

$$\begin{aligned} &d(f\alpha, f\beta) = d(\alpha, \beta) = 2, \\ &d(f\alpha, f\gamma) = d(f\alpha, f\delta) = d(\alpha, \alpha) = 0, \\ &d(f\beta, f\gamma) = d(f\beta, f\delta) = d(\beta, \alpha) = 2, \\ &d(f\gamma, f\delta) = d(\alpha, \alpha) = 0. \end{aligned} \tag{29}$$

Then

$$\begin{aligned} &\frac{6}{5} d(f\alpha, f\beta) = \frac{12}{5} < 8 = \frac{d(\alpha, f\beta) + d(\beta, f\alpha)}{d(\alpha, f\alpha) + d(\beta, f\beta) + 1} d(\alpha, \beta), \\ &\frac{6}{5} d(f\beta, f\gamma) = \frac{12}{5} < 12 = \frac{d(\beta, f\gamma) + d(\gamma, f\beta)}{d(\beta, f\beta) + d(\gamma, f\gamma) + 1} d(\beta, \gamma), \\ &\frac{6}{5} d(f\beta, f\delta) = \frac{12}{5} < 5 = \frac{d(\beta, f\delta) + d(\delta, f\beta)}{d(\beta, f\beta) + d(\delta, f\delta) + 1} d(\beta, \delta). \end{aligned} \tag{30}$$

Thus all the hypotheses of Theorem 11 are satisfied. In this case f has two fixed points $x = \alpha, \beta$ and $d(\alpha, \beta) \geq s/2$. Clearly, f is not a contraction.

Example 15. Let $X = [0, 1]$ and $d : X \times X \to \mathbb{R}_+$ defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 + (x - y)^2 & \text{if } x \neq y. \end{cases} \tag{31}$$

Define $f : X \to X$ by $fx = x$. Clearly, $(X, d, 2)$ is a complete b-metric space. For all $x, y \in X$ with $x \neq y$, we have

$$\begin{aligned} s d(fx, fy) &= 2\left(1 + (x - y)^2\right) \\ &\leq \left(1 + (x - y)^2 + 1 + (x - y)^2\right)\left(1 + (x - y)^2\right) \\ &= \frac{d(x, fy) + d(y, fx)}{d(x, fx) + d(y, fy) + 1} d(x, y). \end{aligned} \tag{32}$$

Then all the hypotheses of Theorem 11 are satisfied. In this case f has infinite fixed points. Clearly, f is not a contraction.

4. Fixed Points for Multivalued Mappings in b-Metric Spaces

In this section, we give a result of existence of fixed point for a class of multivalued mappings.

Theorem 16. *Let (X,d,s) be a complete b-metric space and let $T : X \to CB(X)$ be a multivalued mapping such that*

$$sH(Tx,Ty) \le \frac{d(x,Ty)+d(y,Tx)}{d(x,Tx)+d(y,Ty)+3a}d(x,y) \tag{33}$$

for all $x,y \in X$, where a is a positive real number. Then T has a fixed point $z \in X$.

Proof. Let $x_0 \in X$ be arbitrary and $x_1 \in Tx_0$ such that $d(x_0,x_1) \le d(x_0,Tx_0)+a$. Clearly, if $x_0 = x_1$ or $x_1 \in Tx_1$, we deduce that x_1 is a fixed point of T and so we can conclude the proof. Now, we assume that $x_0 \ne x_1$ and $x_1 \notin Tx_1$ and hence $d(x_1,Tx_1) > 0$. This implies that $H(Tx_0,Tx_1) > 0$. Now, we set

$$h_1 = \sqrt{\frac{d(x_0,Tx_1)}{d(x_0,Tx_0)+d(x_1,Tx_1)+3a}} < 1. \tag{34}$$

By Lemma 8 one can choose $x_2 \in Tx_1$ such that

$$d(x_1,x_2) \le \min\left\{d(x_1,Tx_1)+a, \frac{1}{h_1}H(Tx_0,Tx_1)\right\}. \tag{35}$$

Then

$$\begin{aligned} sd(x_1,x_2) &\le \frac{s}{h_1}H(Tx_0,Tx_1) \\ &\le \frac{1}{h_1}\frac{d(x_0,Tx_1)}{d(x_0,Tx_0)+d(x_1,Tx_1)+3a}d(x_0,x_1) \\ &\le \left(\frac{d(x_0,Tx_1)}{d(x_0,Tx_0)+d(x_1,Tx_1)+3a}\right)^{1/2} d(x_0,x_1) \\ &\le \left(\frac{d(x_0,x_2)}{d(x_0,x_1)+d(x_1,x_2)+a}\right)^{1/2} d(x_0,x_1) \\ &\le s\left(\frac{d(x_0,x_1)+d(x_1,x_2)}{d(x_0,x_1)+d(x_1,x_2)+a}\right)^{1/2} d(x_0,x_1). \end{aligned} \tag{36}$$

This implies

$$d(x_1,x_2) \le \left(\frac{d(x_0,x_1)+d(x_1,x_2)}{d(x_0,x_1)+d(x_1,x_2)+a}\right)^{1/2} d(x_0,x_1). \tag{37}$$

Now, we suppose to have chosen $x_1,\dots,x_n \in X$ such that $x_{i+1} \in Tx_i$, $x_i \notin Tx_i$, and

$$d(x_i,x_{i+1}) \le \left(\frac{d(x_{i-1},x_i)+d(x_i,x_{i+1})}{d(x_{i-1},x_i)+d(x_i,x_{i+1})+a}\right)^{1/2} d(x_{i-1},x_i), \quad \forall i = 1,\dots,n-1. \tag{38}$$

We set

$$h_n = \sqrt{\frac{d(x_{n-1},Tx_n)}{d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)+3a}} < 1. \tag{39}$$

Again, by Lemma 8, one can choose $x_{n+1} \in Tx_n$ such that

$$d(x_n,x_{n+1}) \le \min\left\{d(x_n,Tx_n)+a, \frac{1}{h_n}H(Tx_{n-1},Tx_n)\right\}. \tag{40}$$

Then

$$\begin{aligned} &sd(x_n,x_{n+1}) \\ &\quad\le \frac{s}{h_n}H(Tx_{n-1},Tx_n) \\ &\quad\le \frac{1}{h_n}\frac{d(x_{n-1},Tx_n)}{d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)+3a}d(x_{n-1},x_n) \\ &\quad\le s\left(\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{d(x_{n-1},x_n)+d(x_n,x_{n+1})+a}\right)^{1/2} d(x_{n-1},x_n). \end{aligned} \tag{41}$$

From this, we get

$$\begin{aligned} &d(x_n,x_{n+1}) \\ &\quad\le \left(\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{d(x_{n-1},x_n)+d(x_n,x_{n+1})+a}\right)^{1/2} d(x_{n-1},x_n). \end{aligned} \tag{42}$$

If $x_{n+1} = x_n$, then x_n is a fixed point of T and the proof is finished. If $x_n \notin Tx_n$, itering this procedure we construct a sequence $\{x_n\} \subset X$ such that $x_{n+1} \in Tx_n$, $x_n \notin Tx_n$, and (42) holds for all $n \in \mathbb{N}$. Then, by Proposition 10, $\{x_n\}$ is a Cauchy sequence. Since X is a complete b-metric space, the sequence $\{x_n\}$ converges to some $z \in X$. Now, we prove that z is a fixed point of T. Using (33) with $x = x_n$ and $y = z$, we obtain

$$\begin{aligned} &d(z,Tz) \\ &\quad\le sd(z,x_{n+1})+sd(x_{n+1},Tz) \\ &\quad\le sd(z,x_{n+1})+sH(Tx_n,Tz) \\ &\quad\le sd(z,x_{n+1})+\frac{d(x_n,Tz)+d(z,Tx_n)}{d(x_n,Tx_n)+d(z,Tz)+3a}d(x_n,z) \\ &\quad\le sd(z,x_{n+1})+\frac{sd(x_n,z)+sd(z,Tz)+d(z,x_{n+1})}{d(x_n,x_{n+1})+d(z,Tz)+2a} \\ &\qquad\cdot d(x_n,z). \end{aligned} \tag{43}$$

On taking limit as $n \to +\infty$ on both sides, we get $d(z, Tz) = 0$. As Tz is closed, by Lemma 9, we obtain that $z \in Tz$; that is, z is a fixed point of T. □

Proceeding as in the proof of Theorem 16, we establish the following theorem.

Theorem 17. *Let (X, d, s) be a complete b-metric space and let $T : X \to CB(X)$ be a multivalued mapping such that*

$$sH(Tx, Ty) \leq \frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 3a} d(x, y) + Ld(y, Tx) \quad (44)$$

for all $x, y \in X$, where a is a positive real number and L a nonnegative real number. Then T has a fixed point $z \in X$.

Example 18. Let $X = \mathbb{R}_+$ and $d : X \times X \to \mathbb{R}_+$ defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 2 + (x - y)^2 & \text{if } x \neq y. \end{cases} \quad (45)$$

Define $T : X \to CB(X)$ by $Tx = [x, x + 1]$ for all $x \in X$. Clearly, $(X, d, 2)$ is a complete b-metric space. For all $x, y \in X$ with $x < y$, we have

$$\begin{aligned} sH(Tx, Ty) &= 2\left(2 + (x - y)^2\right) \\ &\leq \left(2 + (x - y)^2\right)\left(2 + (x - y)^2\right) \\ &\leq \frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 1} d(x, y). \end{aligned} \quad (46)$$

Then all the hypotheses of Theorem 16 are satisfied with $a = 1/3$. In this case T has infinite fixed points. Clearly, T is not a multivalued contraction.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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